SCHOOL ON THE MATHEMATICS OF STRING THEORY

April 11 - 15, 2016

Amir-Kian Kashani-Poor: Introduction

This school consists of an array of courses which at first glance may seem to have little in common. The underlying structure relating gauge theory to enumerative geometry to number theory is string theory. In this short introduction, we will attempt to give a schematic overview of how the various topics covered in this school fit into this overarching framework.

MINI-COURSES

Lothar Göttsche: Refined curve counting.

We introduce the Severi degrees of algebraic surfaces, which count nodal curves in linear systems on surfaces. We relate their count to Euler numbers of relative Hilbert schemes. This could be viewed as a mathematical version of BPS state counting. We define two closely related versions of refined curve counting invariants: one is by using instead $\chi_y - genus$ of the relative Hilbert schemes, and could thus be viewed as refined BPS state counting, the other one is via tropical geometry. These invariants are polynomials in a variable y that interpolate between the Severi degrees and the Welschinger invariants, which could real curves. Under suitable assumptions both versions of refined invariants are conjectured to coincide. We will study the generating functions for these invariants, show that they can be computed in terms of a Heisenberg algebra action. Time permitting, we will talk about more recent developments: refined descendent invariants, that interpolate between more general Welschinger invariants and Gromov-Witten invariants with descendants, work of Mikhalkin relating the refined invariants to quantum invariants in real algebraic geometry, and recent attempts to understand the refined Severi degrees in terms of nonarchimedian motivic integration.

Valery Gritsenko: Automorphic forms and Lorentzian Kac-Moody Lie algebras.

The group of symmetries of an affine Kac-Moody algebra is an affine Weyl group. The denominator function of an affine Kac-Moody algebra is a Jacobi modular form. Its modular group is a Jacobi group which is the semidirect product of the Weyl group of a positive definite root system and the Heisenberg group of a root lattice. A Lorentzian (hyperbolic) Kac-Moody Lie algebra has a huge group of hidden symmetries which is much larger than the hyperbolic Weyl group because the denominator function of such algebra is a modular form with respect to an orthogonal group of rank 2. In this mini-course we give an introduction in the theory of automorphic forms and Lorentzian Kac-Moody algebras.

The plan of the lectures is the following:

1. Reflective modular forms on $O(2,n).SL_2$ as orthogonal group.

2. Borcherds modular form Phi_{12} and the Fake Monster Lie algebra.

3. Jacobi forms, the Jacobi lifting construction and Borcherds automorphic products.

4. A new large class of reflective modular forms and the Gritsenko-Nikulin classification of "2-reflective" Lorentzian Kac-Moody Lie algebras.

Sergei Gukov: Homological Algebra of BPS States

In these lectures, we explore BPS spectra of various physical systems. Much like the sound spectrum of a drum contains information about its shape, the BPS spectrum can tell us a great deal about geometry and physics of the underlying physical system. In some cases, BPS spectra turn out to be interesting homological invariants, thus providing connections between math and physics. Moonshine phenomena and physical realizations of knot homologies are good examples of such relations and a rich arena for concrete and explicit computations of BPS spectra. Given its homological nature, a BPS spectrum comes equipped with rich structure and operations common in homological algebra that we dissect in various concrete examples. The main references for these lectures will be:

http://arxiv.org/pdf/1112.0030.pdf

http://arxiv.org/pdf/1211.6075.pdf

http://arxiv.org/pdf/1512.07883.pdf

Jonathan Heckman : 6D SCFTs from String Compactification.

In this set of lectures we provide an overview to the construction and study of six-dimensional superconformal field theories (SCFTs) via compactifications of F-theory on elliptically fibered Calabi-Yau threefolds. 6D SCFTs play an increasingly important role in deducing physical and mathematical structures of higher-dimensional quantum systems as well as lower-dimensional systems. The aim of these lectures will be to explain the general structure of possibly all 6D SCFTs (i.e. to classify them). We explain how this reduces to a geometric problem on the classification of certain elliptic Calabi-Yaus which admit singular degenerations to conformal fixed points. We introduce the basic building blocks of 6D SCFTs and also summarize the classification of such 6D SCFTs (i.e. the corresponding Calabi-Yau geometries). We also explain recent results on understanding RG flows which can connect these theories. A rough outline of what will be covered is:

I: What is a 6D SCFT? (1 hour) II: F-theory in 10D, 8D, 6D (1 hour) III: Classification (1 hour) IV: Renormalization Group Flows (1 hour)

Shamit Kachru : Moonshine and Gravity.

In this elementary series of lectures, I will begin by introducing the objects and ideas involved in Monstrous moonshine. I then describe the elliptic genus of K3 sigma models and its role in uncovering the Mathieu and Umbral moonshines. A third lecture will focus on the intersection of this circle of ideas with approaches to 3d quantum gravity. The fourth and final lecture will discuss how one can use the theory of Jacobi forms to place precise constraints on which two-dimensional supersymmetric field theories have gravitational duals, in the sense of AdS/CFT.

David Morrison : Calabi–Yau manifolds, Mirror Symmetry, and F-theory.

There are five superstring theories, all formulated in 9+1 spacetime dimensions; lower-dimensional theories are studied by taking some of the spatial dimensions to be compact (and small). One of the remarkable features of this setup is that the same lower-dimensional theory can often be realized by pairing different superstring theories with different geometries. The focus of these lectures will be on the mathematical implications of some of these physical "dualities."

Our main focus from the string theory side will be the superstring theories known as type IIA and type IIB. The duality phenomenon occurs for compact spaces of various dimensions and types. We will begin by discussing "T-duality" which uses tori as the compact spaces. We will then digress to introduce M-theory as a strong-coupling limit of the type IIA string theory, and F-theory as a variant of the type IIB string theory whose existence is motivated by T-duality. The next topic is compactifying the type IIA and IIB string theories on K3 surfaces (where the duality involves a change of geometric parameters but not a change of string theory).

By the third lecture, we will have turned our attention to Calabi-Yau manifolds of higher dimension, and the "mirror symmetry" which relates pairs of them. Various aspects of mirror symmetry have various mathematical implications, and we will explain how these are conjecturally related to each other.